# Some topics at the interface of number theory, probability, combinatorics, and analysis

Aled Walker

Tuesday November $8^{th}$ 2022

# Plan

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Lunchtime talk = three-course meal

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**Starter**
Sets of integers with large GCDs (≈ *combinatorial*, 10 mins);

**Main course**
Local distribution of point sequences (≈ *probabilistic*, 20 mins);

**Dessert**
Distribution of primes in short intervals, and zeros of $\zeta(s)$ (≈ *number-theoretic*, 10 mins);

# Warm-up

## Question 1

Suppose $A \subset [1, X]$ is a set of natural numbers with $\gcd(a, a') \geqslant D$ for every pair $(a, a') \in A \times A$. How large can $|A|$ be?

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Suppose $A \subset [1, X]$ is a set of natural numbers with $\gcd(a, a') \geqslant D$ for every pair $(a, a') \in A \times A$. Then $|A| = O(X/D)$.

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**Solution** (Z. Chase): Conjecture is true (pigeonhole).

# The 'real' question

**Question 2**

Let $\delta > 0$, and suppose $A \subset [1, X]$ is a set of natural numbers with $\gcd(a, a') \geqslant D$ for $\delta|A|^2$ pairs $(a, a') \in A \times A$. Is it true that $|A| = O_\delta(X/D)$?

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- study structure of the bipartite graph $(A \times A, \mathcal{E})$, where $(a, a') \in E$ if $\gcd(a, a') \geqslant D$;
- for each prime $p$, iterate according to whether $p|a$ and $p|a'$ (4 subgraphs...).

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**Question 3, Koukoulopoulos–Maynard**

Suppose that $A \subset [1, X]$ is a set of natural numbers with $\gcd(a, a') \geqslant D$ for $0.01|A|^2$ pairs $(a, a') \in A \times A$ and $|A| \gg X/D$. Must there exist some $d$ with $d \gg D$ and $|\{a \in A : d|a\}| \gg X/D$?

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**Answer** (Chow): No

$$A = \{n!/j : n/2 \leqslant j \leqslant n\}.$$

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$\forall \alpha \notin \mathbb{Q}$, $\exists$ *infinitely many pairs* $(a, q)$ *coprime s.t.* $|\alpha - \frac{a}{q}| \leqslant \frac{1}{q^2}$.

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**Theorem (Koukoulopoulos–Maynard, 2020)**

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$$\textit{Then} \quad \mathrm{meas}(\Omega_\psi) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} \frac{\varphi(q)\psi(q)}{q} < \infty \\ 1 & \text{if } \sum_{q=1}^{\infty} \frac{\varphi(q)\psi(q)}{q} = \infty. \end{cases}$$

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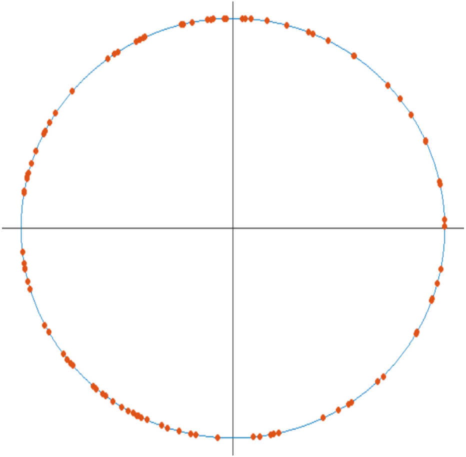
**Proof method**: analysis of GCD structure of $\mathrm{supp}(\psi)$.

# Setting the scene: points on the unit circle

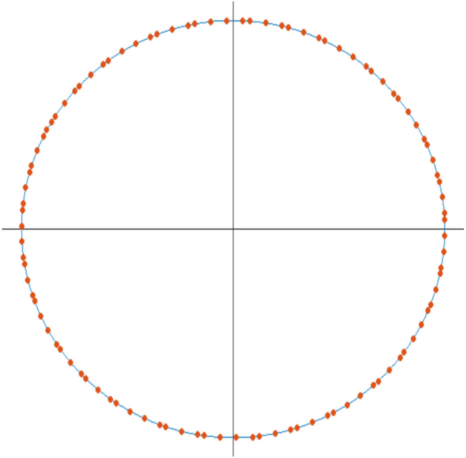
100 independent uniformly random points



# Setting the scene: points on the unit circle

Kronecker sequence

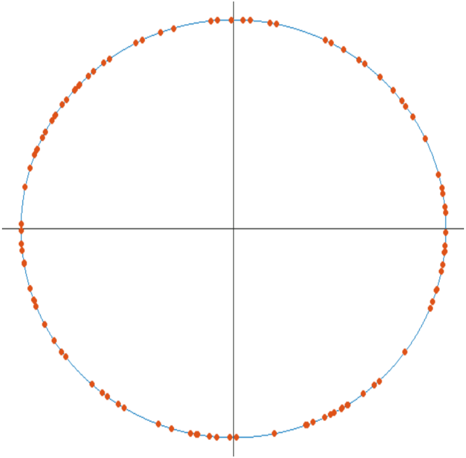
$e^{2\pi i n\sqrt{2}}$, where $n = 1, 2, \ldots, 100$



# Setting the scene: points on the unit circle

'Vinogradov sequence'

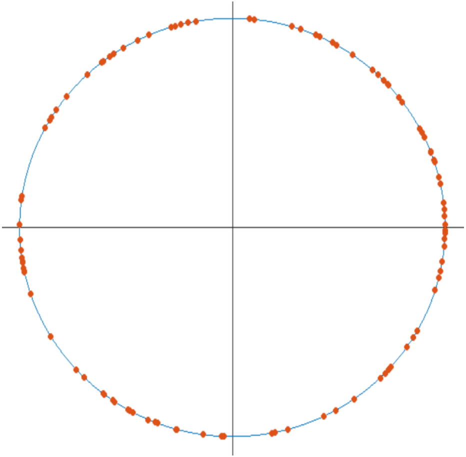
$e^{2\pi i p_n \sqrt{2}}$, where $p_n = n^{th}$ prime, $n = 1, 2, \ldots, 100$



# Setting the scene: points on the unit circle

'Weyl sequence'

$e^{2\pi i n^2 \sqrt{2}}$, where $n = 1, 2, \ldots, 100$



# Consecutive gaps

Given $x_1, x_2, x_3, \ldots \in \mathbb{R}/\mathbb{Z}$, let

$$0 \leqslant \beta_1 < \beta_2 < \cdots < \beta_N < 1$$

be first $N$ elements, in order.

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Consider the measure

$$\mu_{k,N} := \frac{1}{N} \sum_{i \leqslant N} \delta_{N(\beta_{i+k} - \beta_i)} \qquad (\delta_s = \text{Dirac mass at } s).$$

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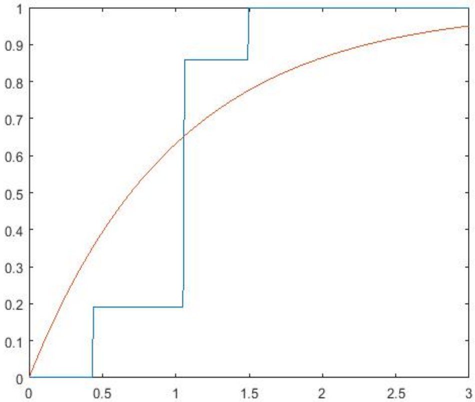
Say $(x_n)_{n=1}^{\infty}$ has *poissonian gaps* if for all $k \geqslant 1$,

$$\mu_{k,N} \xrightarrow{w} \frac{e^{-x} x^{k-1}}{(k-1)!} 1_{x \geqslant 0}\, dx \qquad \text{as } N \to \infty.$$

# Data

$$x_n = n\sqrt{2} \bmod 1, \qquad N = 100000.$$

Plot of d.f. of $\mu_{1,N}$ and of $e^{-x}1_{x\geqslant 0}dx$.




Three-gaps theorem of Sós, Surányi, Świerczkowski (1950s)

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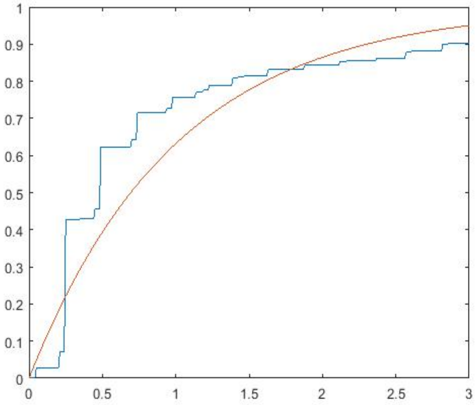
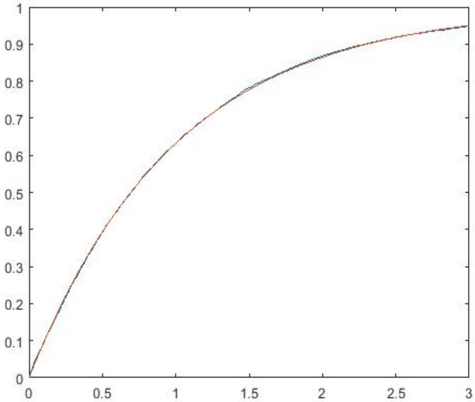


# Data

$$x_n = n^2\sqrt{2} \bmod 1, \qquad N = 100000.$$

Plot of d.f. of $\mu_{1,N}$ and of $e^{-x}1_{x\geqslant 0}dx$.



# Central conjecture

**Conjecture (Rudnick–Sarnak, 1998)**

*For almost all $\alpha \in \mathbb{R}$, the sequence $(\alpha n^2 \bmod 1)_{n=1}^{\infty}$ has poissonian gaps.*

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- motivated by conjectures in physics (gap distribution of eigenvalues of Hamiltonians, Berry–Tabor)
- perhaps very challenging!

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**Pair correlation function**: given $f \in C_c^\infty(\mathbb{R})$, $(a_n)_{n=1}^\infty$ natural numbers

$$R_2(\alpha, L, N, f) := \frac{1}{N} \sum_{\substack{i,j \leqslant N \\ i \neq j}} f\Big(\frac{N}{L}(\alpha(a_i - a_j) \bmod 1)\Big).$$

- $L = N \longrightarrow$ "equidistribution range"
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$(a_n)_{n=1}^\infty$ has **metric poissonian pair correlations** if for almost all $\alpha \in \mathbb{R}$ and for all $f \in C_c^\infty(\mathbb{R})$

$$R_2(\alpha, 1, N, f) \to \widehat{f}(0) \qquad \text{as } N \to \infty.$$

# Results

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**Theorem (W. 2018)**

*The primes do not have metric poissonian pair correlations.*

- in fact, for most $\alpha$, one has $R_2(\alpha, 1, N, f) \gg \log\log N$ infinitely often.

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**Theorem (Bloom–W. 2020)**

*For all increasing sequences $(a_n)_{n=1}^{\infty}$ of natural numbers, $(a_n^2)_{n=1}^{\infty}$ has metric poissonian pair correlations.*

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- $\alpha(p_{1,k} - p_{2,k})$ all contribute to pair correlation function
- hence pair correlation function is large.

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- nature of the arithmetic content?

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What about clustering of the zeros?

# Montgomery's pair correlation conjecture II

For fixed $\beta > 0$, can we get asymptotics for

$$\left(\frac{T}{2\pi}\log T\right)^{-1}\#\gamma,\gamma' \in [0,T] \text{ distinct s.t. } 0 < \gamma - \gamma' \leqslant \frac{2\pi\beta}{\log T}?$$

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Montgomery (1973) conjectured that

$$N_\beta(T) \sim \int_0^\beta 1 - \left(\frac{\sin \pi u}{\pi u}\right)^2 du$$

as $T \to \infty$.

# Odlyzko's data (1987)

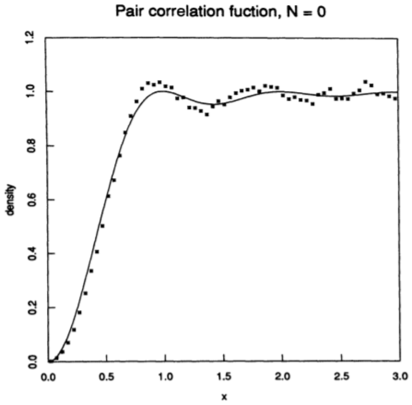


FIGURE 1

*Pair correlation of zeros of the zeta function. Solid line: GUE prediction. Scatter plot: empirical data based on zeros* $\gamma_n$, $1 \leqslant n \leqslant 10^5$.

# Variance of primes

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**Theorem (Goldston-Montgomery '84)**

*Assume RH. Then Montgomery's conjecture is equivalent to the following. For all $\varepsilon > 0$, and for all $h \leqslant X^{1-\varepsilon}$,*

$$\int_0^X (\psi(x+h) - \psi(x) - h)^2 \, dx = (1 + o_\varepsilon(1)) h X \log(X/h)$$

*as $X \to \infty$.*

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**Theorem (observed by Montgomery, later work of Bogomolny, Bolanz, Chan, Goldston, Keating, Soundararajan...)**

*Assume RH. Let* $\varepsilon > 0$. *Suppose that there is some* $\eta \in [1/2, 1)$ *such that for all* $1 \leqslant k \leqslant X^{1-\varepsilon}$ *we have*

$$\sum_{n \leqslant X-k} \Lambda(n)\Lambda(n+k) = \mathfrak{S}(k)(X-k) + O_\varepsilon(X^{\eta+\varepsilon}),$$

*where* $\mathfrak{S}(k)$ *is a certain explicit function of* $k$. *Then for all* $h \leqslant X^{1-\eta}$,

$$\int_0^X (\psi(x+h) - \psi(x) - h)^2 \, dx = (1 + o_\varepsilon(1)) hX \log(X/h)$$

*as* $X \to \infty$.

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**Theorem (Goldston–Gonek–Özlük–Snyder 2000)**

*Assume GRH for Dirichlet L-functions. Let* $\varepsilon > 0$*. Then for all* $h \leqslant X^{1/3-\varepsilon}$,

$$\int_0^\infty (\psi(x+h) - \psi(x) - h)^2 \, dx \geqslant \Big(\frac{1}{2} - o_\varepsilon(1)\Big) hX \log(X/h^3).$$

*(Goldston–Yildrim)*

Under stronger (but standard) assumptions on the distribution of primes in arithmetic progressions, one can improve this.

# Lower bounds

**Theorem (W. 2021+)**

*Suppose that*

$$\psi(x;q,a) = 1_{(a,q)=1}\frac{x}{\varphi(q)} + O_\varepsilon\Big(\frac{x^{1/2+\varepsilon}}{q^{1/2}}\Big)$$

*for all* $x \geqslant 1$, $q < x^{27/53}$, *and* $a \leqslant q$.

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*for all* $x \geqslant 1$, $q < x^{27/53}$, *and* $a \leqslant q$. *Then, for all* $\varepsilon > 0$, *for all* $h \leqslant X^{\frac{1}{95}-\varepsilon}$,

$$\int_0^X (\psi(x+h)-\psi(x)-h)^2\,dx \geqslant \Big(\frac{27}{53} - o_\varepsilon(1)\Big) hX \log\Big(\frac{X}{h^{\frac{127}{27}}}\Big).$$

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Use the fact that divisors $d$ are restricted to $d \leqslant Q$.

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Thank you for your attention!